Abstract: The exact analytic solution of a new boundary value two-dimensional problem is given. This problem considers a partially covered cavity, or trench, located at the intersection of two perpendicular metallic walls. The materials within the cavity and outside are isorefractive to each other. Solutions are given for an electric or magnetic line source located inside the cavity. Numerical results are presented.

1. INTRODUCTION
This boundary-value problem, whose geometry is shown in Fig. 1(a), is solved exactly in the frequency domain, when the primary source is an electric or magnetic line source parallel to the walls and located inside the trench. The field components are expanded in infinite series of Mathieu functions, in the Stratton-Chu notation [1],[2]. The technique utilized in obtaining this new canonical solution is akin to that used for a slotted semielliptical channel [3], that led to numerical results in excellent agreement with integral equation approaches [4], [5]. Some preliminary results for E-polarized plane wave incidence were reported previously in [6]. The canonical solutions presented herein are the only exact, analytical solutions available for a trench located at the corner of two walls. Aside from the intrinsic importance of solving exactly a complicated boundary-value problem involving a cavity, a sharp edge and two different penetrable materials, these canonical solutions constitute a benchmark for the validation of frequency-domain computer codes. Some numerical results are presented. The time-dependence factor \( \exp(\pm j\omega t) \) is omitted throughout.

2. ANALYSIS FOR A LINE SOURCE INSIDE THE CHANNEL
A cross-sectional view of the structure in a plane \( z = \) constant is shown in Fig. 1(a). The trench is a quarter ellipse with interfocal distance \( d \), semi-major axis \( OC = d\xi_1/2 \) and semi-minor axis \( OB = (d/2)\sqrt{\xi_1^2-1} \). The wall \( OE \) is slotted along the slit of width \( OD \) equal to half the interfocal distance \( d \) of the elliptical trench. The trench is partially covered by the thin metal baffle \( DC \). The rectangular coordinates \( (x, y, z) \) are related to the elliptic cylinder coordinates \( (u, v, z) \) by \( x = d/2 \cosh u \cos v, y = d/2 \sinh u \sin v, z = z \), where \( 0 \leq u < \infty, 0 \leq v \leq 2\pi \), \( -\infty < z < \infty \). It is expedient to introduce the coordinates \( \xi = \cosh u, \eta = \cos v \) with \( 1 \leq \xi < \infty \) and \( -1 \leq \eta \leq 1 \). The two media are isorefractive, i.e. \( \varepsilon_1\mu_1 = \varepsilon_2\mu_2 \) so that the propagation constant \( k = \omega\sqrt{\varepsilon_1\mu_1} \) is the same for both media, whilst the intrinsic impedances \( Z_h = \sqrt{\mu_h/\varepsilon_h} \) and \( h = 1, 2 \) are, in general, different from each

Figure 1: 
(a) Geometry
(b) Magnitude of the electric field along the hyperbola \( \eta = 1/2 \) due to an electric line source located at \( (\xi_0 = 1.5, v_0 = -5\pi/12) \) when \( \zeta = 1/2 \). The results correspond to \( c = 1 \) (dotted line), \( c = \pi \) (solid line), and \( c = 10 \) (dash-dot line).
other. The isorefractive condition is necessary for one-to-one mode matching of the fields inside and outside the trench, and the consequent analytical determination of the modal coefficients. Finally, in the following it is convenient to introduce the dimensionless parameter \( c = kd/2 = \pi d/\lambda \), where \( \lambda \) is the wavelength. Let us consider an electric line source \( E^1_x = \frac{\pi}{2} H^\perp (kR) \) located inside the trench at \((u_0, v_0) = (\xi_0, \eta_0)\), i.e., \( 1 \leq \xi_0 < \xi_1 \) and \( 3\pi/2 < v_0 < 2\pi \). The total electric field inside the trench is \( E_{2x} = E^1_x + E^2_{2x} \). The geometric-optics field \( E^1_x \) is given by \( E^1_x = 16 \sum_{l=1}^{\infty} \frac{1}{N_{2l}} \text{Ro}^{(1)}_{2l}(c, \xi_0) \text{Ro}^{(4)}_{2l}(c, \xi_0) \text{So}_{2l}(c, \xi_0) \text{So}_{2l}(c, \eta) \) and represents the total field that would be present in the absence of the trench, i.e. the sum of the fields due to four line sources: the primary line and its three images. The diffracted field component \( E^2_{2x} \) accounts for the slot at \( \xi \) = 1 as well as the elliptic metal wall at \( \xi = \xi_1 \), and may be written as \( E^2_{2x} = 16 \sum_{l=0}^{\infty} \frac{1}{N_{2l}} \left[ \tilde{a}_l \text{Ro}^{(1)}_{2l}(c, \xi) + \tilde{c}_l \text{Ro}^{(4)}_{2l}(c, \xi) \right] \text{So}_{2l}(c, \xi_0) \text{So}_{2l}(c, \eta) \).

The total field \( E_{1x} \) in the quadrant \((x \geq 0, y \geq 0)\) is \( E_{1x} = -16 \sum_{l=0}^{\infty} \frac{1}{N_{2l}} \text{Ro}^{(4)}_{2l}(c, \xi_0) \text{So}_{2l}(c, \xi_0) \text{So}_{2l}(c, \eta) \), and the modal coefficients \( \tilde{a}_l \) and \( \tilde{c}_l \) are explicitly determined by imposing the boundary conditions:

\[
\tilde{a}_l = - \left( \text{Ro}^{(4)}_{2l}(c, \xi_1)/\Delta_l^{(0)} \right) \left[ \text{Ro}^{(1)}_{2l}(c, 1) \text{Ro}^{(4)}_{2l}(c, \xi_0) - (1 + \zeta^{-1}) \text{Ro}^{(4)}_{2l}(c, 1) \text{Ro}^{(1)}_{2l}(c, \xi_0) \right],
\]

\[
\tilde{c}_l = - \left( \text{Ro}^{(1)}_{2l}(c, 1)/\Delta_l^{(0)} \right) \left[ \text{Ro}^{(1)}_{2l}(c, \xi_0) \text{Ro}^{(4)}_{2l}(c, \xi_1) - \text{Ro}^{(4)}_{2l}(c, \xi_0) \text{Ro}^{(1)}_{2l}(c, \xi_1) \right],
\]

where \( \Delta_l^{(0)} = \text{Ro}^{(1)}_{2l}(c, 1) \text{Ro}^{(4)}_{2l}(c, \xi_1) - (1 + \zeta^{-1}) \text{Ro}^{(4)}_{2l}(c, 1) \text{Ro}^{(1)}_{2l}(c, \xi_1) \). The surface current densities on the metal boundaries are given by \( J_z = -H_{1x} \) on \( DE \) \((v = 0)\), \( J_z = H_{2x} \) on \( DC \) \((v = 2\pi)\), \( J_z = H_{1x} \) on \( OA \) \((v = \pi/2)\), \( J_z = -H_{2x} \) on \( OB \) \((v = 3\pi/2)\), \( J_z = -H_{2x} \) on \( BC \) \((\xi = \xi_1)\), where the magnetic field is obtained from \( H = \left( j/(cZ) \left( \hat{a}_E \hat{u} - \hat{a}_H \hat{v} \right) \right) \). Since the derivations for H-polarization are similar to those for E-polarization they are not given here for brevity.

4. NUMERICAL RESULTS AND CONCLUSION

The computation of the Mathieu functions was performed using some of the Fortran subroutines provided in [7]. However, these subroutines are based on the Golstein-Ince normalization so they were appropriately modified for the normalization used in this work. Computations were performed for \( \xi_1 = 2 \) and \( c = 1, \pi, 10 \). Fig. 1(b), shows \(|E_x|\) due to an electric line located inside the trench. This geometry may be used to examine the field radiated by a wire that is located inside the trench of Fig. 1(a). The strongest field contribution is observed when \( c = \pi \), i.e. \( d/\lambda = 1 \), which may suggest the existence of a resonance condition that causes a better coupling with the field outside the trench. Figs. 2(a) and 2(b) provide results for \(|J_z|\) induced on the metallic walls by an electric or magnetic line source, respectively. Both results were computed with the help of the Shanks transform acceleration method applied only to the imaginary part of the terms of the series involved, similar to what is described in [8]. The geometrical optics terms were evaluated using the direct expressions in terms of Hankel functions.

Exact analytical solutions have been derived for a line source located inside a slotted quarter-elliptical trench in a corner. These new canonical solutions are important not only per se, but also because they provide benchmarks for the validation of frequency-domain codes involving geometries with trenches, sharp edges, and different penetrable materials. Numerical results based on these exact solutions have been obtained and discussed. Further details will be available in [9].

6. ACKNOWLEDGEMENT

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References


(a) Currents induced on the metallic walls of the structure of Fig. 1(a) due to an electric line source located at $(\xi_0 = 1.5, v_0 = -5\pi/12)$ when $\zeta = 1/2$. Current $|J_z|$ along $DE$ is shown in (a); current $|J_z|$ along $OA$ is shown in (b); current $|J_z|$ along $OB$ is shown in (c); current $|J_z|$ along $BC$ is shown in (d); and current $|J_z|$ along $DC$ is shown in (e). The results shown in (a-e) correspond to $c = 1$ (dotted line), $c = \pi$ (solid line), and $c = 10$ (dash-dot line).

(b) Currents induced on the metallic walls of the structure of Fig. 1(a) due to a magnetic line source located at $(\xi_0 = 1.5, v_0 = -5\pi/12)$ when $\zeta = 2$. Current $|J_z|$ along $DE$ is shown in (a); current $|J_z|$ along $OA$ is shown in (b); current $|J_z|$ along $OB$ is shown in (c); current $|J_z|$ along $BC$ is shown in (d); and current $|J_z|$ along $DC$ is shown in (e). The results shown in (a-e) correspond to $c = 1$ (dotted line), $c = \pi$ (solid line), and $c = 10$ (dash-dot line).

Figure 2:


