Exact Analysis of an Oblate Spheroidal Cavity with a Circular Aperture in a Ground Plane Covered by an Isorefractive Lens

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Abstract — An oblate semi-spheroidal cavity with metallic walls flush-mounted under a metallic ground plane, and coupled to the half-space above via its circular interfocal aperture, is separated from the air above by a lens. Both the lens material and the material filling the cavity are isorefractive to free space. An exact solution is obtained for the radiation of an electric or magnetic dipole located on the symmetry axis of the structure either above the lens, or inside the cavity below the lens, and axially oriented. Numerical results are also provided.

1 INTRODUCTION

The exact solution to a three-dimensional electromagnetic boundary-value problem involving an oblate semi-spheroidal cavity with metallic walls flush-mounted under a metallic ground plane and coupled to the half-space above the plane via a circular hole is considered. The material above the ground plane is separated from the material inside the cavity by a lens. All materials are isorefractive to each other.

The primary source is an electric or magnetic dipole located on the axis of symmetry of the structure and axially oriented. The exact solution is written in the form of series expansions involving oblate spheroidal wave functions. The expansion coefficients in the series are determined analytically by imposing the boundary conditions, thereby leading to a canonical solution of the boundary-value problem. The technique is an extension of that used by Berardi et al. [1]. The notation for the spheroidal wave functions is that of Flammer [2].

The exact solution of this complicated problem, which involves a sharp curved metallic edge, a cavity, and a curved surface separating different penetrable media enriches the catalog of available canonical solutions, and provides a challenging test for the validation of frequency-domain codes.

Numerical results based on the evaluation of the series of oblate spheroidal functions are provided for the fields inside the cavity, inside the lens and in the open space above the structure. Issues such as the evaluation of the fields near the edge of the cavity and the effect of the lens presence, are analyzed in detail. The time-dependence factor $\exp(-i\omega t)$ is omitted throughout.

2 GEOMETRY OF THE PROBLEM

The geometry of the problem is symmetric with respect to the $z$ axis and it is shown in cross section in Fig. 1. The various boundaries correspond to coordinate surfaces of a right-handed oblate spheroidal coordinate system $(\eta, \xi, \varphi)$, with foci located at points A and B, whose distance is the interfocal distance $d$. The surface $\eta = 0$ is metallic and corresponds to the plane $z = 0$ without the circular focal disk of diameter $d$. Below the aperture there is a cavity that is limited by a metallic boundary located on an oblate semi-spheroid at $\xi = \xi_1$. Across the aperture is located a lens, which is made of two different isorefractive materials and limited by the surfaces $\xi = \xi_2$ and $\xi = \xi_3$ below and above the aperture, respectively. Both the circular aperture and the lens provide the coupling between the cavity and the unbounded medium.

In general, the surfaces $|\eta| = \text{constant}$ represent hyperboloids of revolution with $z$ as the symmetry axis, $\eta > 0$ ($\eta < 0$) for $z > 0$ ($z < 0$), and asymptotic cone of semi-aperture $\theta = \arccos \eta$; in particular, $\eta = 1$ is the positive $z$-axis, whereas $\eta = -1$ represents the negative $z$-axis (see Fig. 1). The surface $\varphi = \text{constant}$ is a half-plane originating in the $z$ axis. The oblate spheroidal coordinate $(\eta, \xi, \varphi)$ satisfy $0 \leq \xi < \infty$, $-1 \leq \eta \leq 1$ and $0 \leq \varphi \leq 2\pi$. Further details on oblate spheroidal coordinates and their application to a similar geometry are available in [1].

Four different media are considered in this problem, as shown in Fig. 1: the unbounded medium with dielectric permittivity $\varepsilon_1$ and magnetic permeability $\mu_1$; the lens that is made of material with parameters $\varepsilon_3, \mu_3$ for $z > 0$ and $\varepsilon_4, \mu_4$ for $z < 0$; and the cavity with parameters $\varepsilon_2$ and $\mu_2$. Since the media are isorefractive, they have the same propa-
gation constant

\[ k = \omega \sqrt{\varepsilon_0 \mu_0} = \omega \sqrt{\varepsilon_h \mu_h}, \quad h = 1, \ldots, 4; \quad (1) \]

however, in general, they have different impedances

\[ Z_l = Y_l^{-1} = \sqrt{\frac{\mu_h}{\varepsilon_l}} \neq Z_m, \quad l, m = 1, \ldots, 4, \quad l \neq m. \quad (2) \]

![Figure 1: Geometry of the problem. The dashed hyperbolic curves represent trajectories along which the fields are computed.](image)

The primary source is an electric or magnetic dipole located either in medium 1 or in medium 2 on the \( z \) axis and axially oriented (Section 3). Numerical results are presented in Section 4.

3 ANALYTICAL RESULTS

3.1 Electric Dipole in Medium 1

When an electric dipole is located on the \( z \) axis, and axially oriented, the general expressions for the electric and magnetic fields in oblate spheroidal coordinates are [1], [3]:

\[ E = E_\xi(\xi, \eta) \hat{\xi} + E_\eta(\xi, \eta) \hat{\eta}, \quad E_\varphi = 0, \quad H_\xi = H_\eta = 0. \quad (3) \]

An infinitesimal dipole located in medium 1 at \((\xi_0, \eta_0) = 1\) with moment \( \hat{z} \exp(ikR)/(kR) \), where \( R \) is the distance of the observation point \((\eta, \xi, \varphi)\) from the dipole, generates a primary magnetic field in the unbounded medium [3]

\[ H^1_{1,\varphi} = \frac{2k^2Y_1}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-i)^n S_{1,n}(-ic, \eta)}{\tilde{\rho}_{1,n} N_{1,n}} \times R^{(1)}_{1,n}(-ic, i\xi_0) R^{(3)}_{1,n}(-ic, i\xi_0), \quad (4) \]

where \( c = kd/2 \) is the product of the wavenumber and the inter focal radius. Using the notation of Flammer [2], \( Y_{1,n} \) is the angular oblate spheroidal function of order 1 and degree \( n \), \( R^{(1,3)}_{1,n} \) are the radial oblate spheroidal functions of order 1, degree \( n \), and of the first and third kind, and \( \xi_0(\xi_0) \) is the smaller(larger) between \( \xi \) and \( \xi_0 \). The general expression for the field in medium 1 can be written as

\[ H_{1,\varphi} = H^1_{1,\varphi} + H^H_{1,\varphi} + H^d_{1,\varphi}, \quad (5) \]

where \( H^H_{1,\varphi} \) is given by (4), \( H^1_{1,\varphi} \) is the reflected field if the cavity and the lens were removed, and \( H^d_{1,\varphi} \) is the perturbation field due to the presence of both the cavity and the lens. By applying the image theory and recalling some special values of the angular oblate functions (see [1], Appendix) the geometrical optics field in medium 1 is:

\[ H^H_{1,\varphi} = H^1_{1,\varphi} + H^d_{1,\varphi} \]

The radiation condition applied to the diffracted field \( H^d_{1,\varphi} \) in the unbounded medium yields:

\[ H^d_{1,\varphi} = \frac{-4ik^2Y_1}{\sqrt{\xi_0^2 + 1}} \sum_{l=0}^{\infty} \frac{(-1)^l S_{1,2l+1}(-ic, \eta)}{\tilde{\rho}_{1,2l+1} N_{1,2l+1}} \times R^{(1)}_{1,2l+1}(-ic, i\xi_0) R^{(3)}_{1,2l+1}(-ic, i\xi_0). \quad (6) \]

The general expression for the total magnetic field in the other media is given by a linear combination of radial functions of the first and third kind:

\[ H_{1,\varphi}^H = \frac{-4ik^2Y_h}{\sqrt{\xi_h^2 + 1}} \sum_{l=0}^{\infty} \frac{(-1)^l S_{1,2l+1}(-ic, \eta)}{\tilde{\rho}_{1,2l+1} N_{1,2l+1}} \times \left[ a_{l,1}^H R^{(1)}_{1,2l+1}(-ic, i\xi_0) + b_{l,1}^H R^{(3)}_{1,2l+1}(-ic, i\xi_0) \right], \quad (7) \]

where \( h = 3, 4 \).

The coefficients \( a_{l,1}^H, b_{l,1}^H \) are determined imposing the boundary conditions:

\[ \begin{aligned}
  a_{l,1}^H - \frac{R^{(3)}_{1,2l+1}(-ic, i\xi_0) \Delta a_{l,1}^H}{\Delta(e)} \\
  b_{l,1}^H - \frac{-R^{(1)}_{1,2l+1}(-ic, i\xi_0) \Delta b_{l,1}^H}{\Delta(e)}
\end{aligned} \quad (10) \]

with \( r = 1, \ldots, 4, \quad s = 2, \ldots, 4, \quad \) and where:
The contour plot of the magnitude of the electric field due to a magnetic dipole located in medium 1 is shown in Fig. 1. In order to appreciate the effect of the lens and of the cavity, only the perturbation field is considered in medium 1 and 3, whereas the total field is plotted in medium 2 and 4.

The quantities of interest that need to be computed are \( E_\varphi \) or \( H_\varphi \), when either a magnetic or electric source is considered, respectively. They act as scalar potentials, because all the other fields can be easily derived by applying Maxwell’s equations in the oblate spheroidal coordinate system \([1]\). The fields are evaluated along the coordinate lines \(|\eta| = \text{constant}\) of Fig. 1. Several values of the parameter \( c = kd/2 \) are considered because it has the physical meaning of the ratio of the aperture size to the wavelength. In all the numerical results, the curved metallic cavity corresponds to the semi-spheroidal coordinate surface \( \xi_1 = 2 \), whereas the lower and the upper faces of the lens are given by \( \xi_2 = 1 \) and \( \xi_2 = 1.25 \) respectively. Also, the dipole sources in medium 1 are located at \((\xi_0 = 1.5, \eta_0 = 1)\), whereas the ones in medium 2 are at \((\xi_0 = 1.5, \eta_0 = -1)\).

The magnitude of the total magnetic field \(|H_\varphi|\) when the source is an electric dipole located in medium 2 is presented in Fig. 3. In Fig. 4 it is plotted the total electric field \(|E_\varphi|\) when the source is a magnetic dipole in medium 2.

The contour plot of the magnitude of the electric field due to a magnetic dipole located in medium 1 is shown in Fig. 5. In order to appreciate the effect of the lens and of the cavity, only the perturbation field is considered in medium 1 and 3, whereas the total field is plotted in medium 2 and 4.

The analytical derivations of the fields for the other cases, such as a magnetic dipole located in medium 1 at \((\xi_0, \eta_0) = (1,1)\) and an electric or magnetic dipole located inside medium 2 at \((\xi_0, \eta_0) = (-1)\) are similar, but are not reported here for lack of space. However, in Section 4 numerical results are presented for each case.

### 4 NUMERICAL RESULTS

The numerical evaluation of the fields has been performed using some of the Fortran routines that supplement oblate spheroidal radial and angular functions published in \([4]\), and, in order to achieve convergence, the acceleration technique reported in \([5]\).

In the particular case when \( \zeta_{13} = \zeta_{42} = 1 \) and \( \zeta_{34} = \zeta \) the modal coefficients become \([1]\):

\[
a^{(c)}_{41} = a^{(c)}_{41} = a^{(c)}_{31} = a^{(c)}_{31} = 0,
\]

\[
a^{(c)}_{21} = -\frac{R^{(3)}_{1,21+1}(-ic, i\zeta)}{R^{(1)}_{1,21+1}(-ic, i\zeta)} \zeta R^{(1)}_{1,21+1}(-ic, i\zeta) + (1 + \zeta) a^{(c)}_{21} R^{(3)}_{1,21+1}(-ic, i\zeta),
\]

\[
a^{(c)}_{11} = M^{(c)}_{11},
\]

\[
b^{(c)}_{31} = b^{(c)}_{11} = \frac{a^{(c)}_{31}}{M^{(c)}_{11}},
\]

\[
b^{(c)}_{21} = R^{(3)}_{1,21+1}(-ic, i\zeta) / M^{(c)}_{11}.
\]
5 CONCLUSION

Analytical and numerical results for an electromagnetic boundary-value problem involving a cavity, sharp edges, a lens and four isorefractive media were presented. These results are important because they enrich the list of problems for which exact solutions are known and may be used to validate solutions obtained with other methods.

References


